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Optimal Time Splitting for Two- and Three-Dimensional Navier–Stokes Equations with Mixed Derivatives*

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A new explicit, time splitting algorithm has been developed for finite difference modelling of the full two and three-dimensional time-dependent, compressible, viscous Navier–Stokes equations of fluid mechanics. The scheme is optimal in the sense that the split operators achieve their maximum allowable time step, i.e., the corresponding Courant number. The algorithm allows a conservation-form formulation. Stability is proven analytically and verified numerically. In proving stability it was shown that all nine matrix coefficients of the Navier–Stokes equations are simultaneously symmetrizable by a similarity transformation. Two such transformations and their resulting symmetric matrix coefficients are presented explicitly.

I. INTRODUCTION

During the 1970s there has been growing attention paid to the numerical solution of the full two-dimensional, time-dependent, viscous, compressible Navier–Stokes equations of fluid mechanics; see inter alia [1–4]. MacCormack and Baldwin [4] use an explicit time splitting technique, even though the presence of the viscous mixed derivative term (due to compressibility) does not allow the usual split in the respective spatial-coordinate directions. They attempted to overcome this difficulty by apportioning the mixed derivative term among the space-split operators. The resulting difference operators do not constitute a "classical" split in the sense of Strang [5], Marchuk [6] and others, in that the norms of these split operators (none of which are one-dimensional spatially) are not bounded by unity. This complicates the stability analysis of their second order finite difference scheme and therefore they present only an estimated stability condition. It turns out, even as seen from the estimated criterion, that a drawback of their method of splitting is that the time step in each direction (x, y) is restricted by the mesh size in the other direction. In particular, when $\Delta x/\Delta y \gg 1$ (typical, for example, in calculations of boundary layers) the allowed time step, even in the x-direction, diminishes as $\Delta y/\Delta x$.

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In the present paper we propose a new method of splitting which leads, for any explicit second order accurate finite difference scheme, to optimal time steps in all directions. The analysis for the (linear) stability of this algorithm is carried out analytically for the three-dimensional case (from which the results for the two-dimensional case are, of course, found immediately by reduction). This is made possible by the construction of a similarity transformation which symmetrizes *all* the matrix coefficients of the Navier–Stokes equations simultaneously (there are nine such coefficients matrices in the three-dimensional case and five in the two-dimensional case).

In Section II we present the governing three-dimensional fluid dynamic equations both in conservation and non-conservative forms. We show how the nine matrix coefficients of these Navier–Stokes equations may all be symmetrized simultaneously by a similarity transformation.

In Section III we present the new method of splitting, which may be used in conjunction with any finite difference algorithm. We also present one particular three-dimensional explicit algorithm resulting from the repeated application of the MacCormack one-dimensional scheme.

The (linear) stability analysis is carried out fully for the three-dimensional case in Section IV.

In Section V we present numerical evidence to support the conclusions predicted by the analysis.

In Appendix A we summarize the results of Sections II and III for the two-dimensional case. This is done to allow ease of application for those interested primarily in two-dimensional problems.

II. The Navier–Stokes Equations

The time-dependent compressible viscous fluid dynamic equations in three-dimensions, and in the absence of body forces, may be written in conservation form as

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}}{\partial x}+\frac{\partial \mathbf{G}}{\partial y}+\frac{\partial \mathbf{H}}{\partial z}=0, \tag{2.1}$$

where the transpose of the vector U is given by $(\rho, \rho u, \rho v, \rho w, e)$, and

$$\mathbf{F}=\begin{bmatrix} \rho u \\ \rho u^2+\tau_{xx} \\ \rho uv+\tau_{yx} \\ \rho uw+\tau_{zx} \\ (e+\tau_{xx})\,u+\tau_{yx}v+\tau_{zx}w-k\dfrac{\partial T}{\partial x} \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \rho v \\ \rho v u + \tau_{xy} \\ \rho v^2 + \tau_{yy} \\ \rho v w + \tau_{zy} \\ (e + \tau_{yy}) v + \tau_{zy} w + \tau_{xy} u - k \dfrac{\partial T}{\partial y} \end{bmatrix}, \tag{2.2}$$

$$\mathbf{H} = \begin{bmatrix} \rho w \\ \rho w u + \tau_{xz} \\ \rho w v + \tau_{yz} \\ \rho w^2 + \tau_{zz} \\ (e + \tau_{zz}) w + \tau_{xz} u + \tau_{yz} v - k \dfrac{\partial T}{\partial z} \end{bmatrix},$$

and, where

$$\tau_{xx} = p - 2\mu \frac{\partial u}{\partial x} - \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right), \qquad \tau_{xy} = \tau_{yx} = -\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

$$\tau_{yy} = p - 2\mu \frac{\partial v}{\partial y} - \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right), \qquad \tau_{yz} = \tau_{zy} = -\mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),$$

$$\tau_{zz} = p - 2\mu \frac{\partial w}{\partial z} - \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right), \qquad \tau_{zx} = \tau_{xz} = -\mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right).$$

The various parameters and dependent variables ρ, u, v, w, μ, λ, e, k, T and p are respectively the density, the x, y and z components of velocity, the shear and second coefficients of viscosity, the total energy per unit volume, the coefficient of heat conduction, the temperature and pressure. An equation of state $p = p(\varepsilon, \rho)$ relates the pressure to the density and the specific internal energy, $\varepsilon = (e/\rho) - (u^2 + v^2 + w^2)/2$.

In non-conservative form the Navier–Stokes equations for a perfect gas may be written as

$$\frac{\partial \mathbf{V}}{\partial t} + A \frac{\partial \mathbf{V}}{\partial x} + B \frac{\partial \mathbf{V}}{\partial y} + J \frac{\partial \mathbf{V}}{\partial z}$$
$$= C \frac{\partial^2 \mathbf{V}}{\partial x^2} + D \frac{\partial^2 \mathrm{v}}{\partial y^2} + K \frac{\partial^2 \mathbf{V}}{\partial z^2} + E_{xy} \frac{\partial^2 \mathbf{V}}{\partial x \, \partial y} + E_{yz} \frac{\partial^2 \mathbf{V}}{\partial y \, \partial z} + E_{zx} \frac{\partial^2 \mathbf{V}}{\partial z \, \partial x}, \tag{2.3}$$

where $\mathbf{V}$ is the column vector (ρ, u, v, w, p) and the coefficient matrices are given by

$$A = \begin{bmatrix} u & \rho & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 1/\rho \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & 0 & u \end{bmatrix}, \qquad B = \begin{bmatrix} v & 0 & \rho & 0 & 0 \\ 0 & v & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 1/\rho \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & \gamma p & 0 & v \end{bmatrix}, \tag{2.4}$$

$$J = \begin{bmatrix} w & 0 & 0 & \rho & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & w & 1/\rho \\ 0 & 0 & 0 & \gamma p & w \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda + 2\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu}{\rho} & 0 \\ -\frac{\gamma\mu p}{P_r \rho^2} & 0 & 0 & 0 & \frac{\gamma\mu}{P_r \rho} \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda + 2\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu}{\rho} & 0 \\ -\frac{\gamma\mu p}{P_r \rho^2} & 0 & 0 & 0 & \frac{\gamma\mu}{P_r \rho} \end{bmatrix},$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda + 2\mu}{\rho} & 0 \\ -\frac{\gamma\mu p}{P_r \rho^2} & 0 & 0 & 0 & \frac{\gamma\mu}{P_r \rho} \end{bmatrix},$$

$$E_{xy} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\lambda+\mu}{\rho} & 0 & 0 \\ 0 & \dfrac{\lambda+\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_{yz} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\lambda+\mu}{\rho} & 0 \\ 0 & 0 & \dfrac{\lambda+\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad E_{zx} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\lambda+\mu}{\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\lambda+\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where γ is the ratio of specific heats and P_r is the Prandtl Number.

It has been shown in the past [7, 8] that in the *two-dimensional* case the two 4×4 "hyperbolic" matrices (corresponding to A and B in (2.3)) may be symmetrized simultaneously. We were motivated to ask whether all nine coefficient matrices in (2.3) can be symmetrized simultaneously. A systematic way to find out is to start with the realization that if a set of $m \times m$ matrices $A_1,..., A_n$ is to be simultaneously symmetrizable then there must be a similarity transformation S characterized by the following: $S^{-1}A_jS$, for some $1 \leqslant j \leqslant n$ is diagonal and $S^{-1}A_kS$ is symmetric $\forall\ 1 \leqslant k \leqslant n$. If A_j does not have multiple eigenvalues we may then start by constructing a matrix T from the eigenvectors of A_j, and if $A_1,..., A_n$ are indeed simultaneously symmetrizable, then the most general symmetrizer for A_k is $S = T\Lambda$ for some diagonal matrix Λ. The diagonal matrix Λ is then found from the conditions stemming from the requirement that $T^{-1}A_kT\Lambda^2$ be symmetric for all $k \neq j$. Since there are $m(m-1)(n-1)/2$ equations to determine the n elements of Λ, there will be no solution if the set of given matrices is not simultaneously symmetrizable. At first glance it seems that the above procedure will fail since all nine matrix coefficients have multiple eigenvalues. A closer inspection will show however that the matrices C, D and K are "almost" diagonal in that their only non-zero off diagonal term is a corner element while the multiple eigenvalues, μ/ρ, lie in the inner 3×3 minor. Thus we really have to diagonalize only a 2×2 "outer" matrix of the form

$$\begin{bmatrix} 0 & 0 \\ -\dfrac{\gamma\mu p}{P_r\rho^2} & \dfrac{\gamma\mu}{P_r\rho} \end{bmatrix},$$

which has distinct roots. Therefore, the matrix coefficients of the Navier–Stokes equations meet the conditions stated above and hence can be simultaneously symmetrized.

We now present two such symmetrizers. The first is derived by first diagonalizing the matrix A. The resulting similarity transformation is related to the one given in [7, 8] and we designate it S_H to indicate that we start with the hyperbolic portion of the equations.[1]

$$S_H = \begin{bmatrix} \beta\rho & \rho & 0 & 0 & \rho \\ 0 & c & 0 & 0 & -c \\ 0 & 0 & \sqrt{2}c & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}c & 0 \\ 0 & \rho c^2 & 0 & 0 & \rho c^2 \end{bmatrix},$$

$$S_H^{-1} = \begin{bmatrix} \dfrac{1}{\beta\rho} & 0 & 0 & 0 & -\dfrac{1}{\beta\rho c^2} \\ 0 & \dfrac{1}{2c} & 0 & 0 & \dfrac{1}{2\rho c^2} \\ 0 & 0 & \dfrac{1}{\sqrt{2}c} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\sqrt{2}c} & 0 \\ 0 & -\dfrac{1}{2c} & 0 & 0 & \dfrac{1}{2\rho c^2} \end{bmatrix}, \tag{2.5}$$

where $c = (\gamma p/\rho)^{1/2}$ is the speed of sound and $\beta = \sqrt{2(\gamma - 1)}$. For the sake of completeness the symmetrized matrices are now given:

$$S_H^{-1} A S_H = \begin{bmatrix} u & 0 & 0 & 0 & 0 \\ 0 & u+c & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & u-c \end{bmatrix},$$

[1] Note that A does have a triple eigenvalue, u, and yet our procedure worked.

$$S_{\mathrm{H}}^{-1}BS_{\mathrm{H}}=\begin{bmatrix} v & 0 & 0 & 0 & 0 \\ 0 & v & \dfrac{c}{\sqrt{2}} & 0 & 0 \\ 0 & \dfrac{c}{\sqrt{2}} & v & 0 & \dfrac{c}{\sqrt{2}} \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & \dfrac{c}{\sqrt{2}} & 0 & v \end{bmatrix},$$

$$S_{\mathrm{H}}^{-1}JS_{\mathrm{H}}=\begin{bmatrix} w & 0 & 0 & 0 & 0 \\ 0 & w & 0 & \dfrac{c}{\sqrt{2}} & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & \dfrac{c}{\sqrt{2}} & 0 & w & \dfrac{c}{\sqrt{2}} \\ 0 & 0 & 0 & \dfrac{c}{\sqrt{2}} & w \end{bmatrix},$$

$$S_{\mathrm{H}}^{-1}CS_{\mathrm{H}}=\begin{bmatrix} \dfrac{\mu}{P_r\rho} & -\dfrac{\beta\mu}{2P_r\rho} & 0 & 0 & -\dfrac{\beta\mu}{2P_r\rho} \\ -\dfrac{\beta\mu}{2P_r\rho} & \dfrac{\beta^2}{4}\dfrac{\mu}{P_r\rho}+\dfrac{\lambda+2\mu}{2\rho} & 0 & 0 & \dfrac{\beta^2}{4}\dfrac{\mu}{P_r\rho}-\dfrac{\lambda+2\mu}{2\rho} \\ 0 & 0 & \dfrac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\mu}{\rho} & 0 \\ -\dfrac{\beta\mu}{2P_r\rho} & \dfrac{\beta^2}{4}\dfrac{\mu}{P_r\rho}-\dfrac{\lambda+2\mu}{2\rho} & 0 & 0 & \dfrac{\beta^2}{4}\dfrac{\mu}{P_r\rho}+\dfrac{\lambda+2\mu}{2\rho} \end{bmatrix},$$

$$S_{\mathrm{H}}^{-1} D S_{\mathrm{H}} = \begin{bmatrix} \frac{\mu}{P_r \rho} & -\frac{\beta u}{2P_r\rho} & 0 & 0 & -\frac{\beta\mu}{2P_r\rho} \\ -\frac{\beta\mu}{2P_r\rho} & \frac{\beta^2}{4}\frac{\mu}{P_r\rho} + \frac{\mu}{2\rho} & 0 & 0 & \frac{\beta^2}{4}\frac{\mu}{P_r\rho} - \frac{\mu}{2\rho} \\ 0 & 0 & \frac{\lambda + 2\mu}{2\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu}{\rho} & 0 \\ -\frac{\beta\mu}{2P_r\rho} & \frac{\beta^2}{4}\frac{\mu}{P_r\rho} - \frac{\mu}{2\rho} & 0 & 0 & \frac{\beta^2}{4}\frac{\mu}{P_r\rho} + \frac{\mu}{2\rho} \end{bmatrix}, \tag{2.6}$$

$$S_{\mathrm{H}}^{-1} K S_{\mathrm{H}} = \begin{bmatrix} \frac{\mu}{P_r \rho} & -\frac{\beta\mu}{P_r\rho} & 0 & 0 & -\frac{\beta\mu}{2P_r\rho} \\ -\frac{\beta\mu}{P_r\rho} & \frac{\beta^2}{4}\frac{\mu}{P_r\rho} + \frac{\mu}{2\rho} & 0 & 0 & \frac{\beta^2 u}{4P_r\rho} - \frac{\mu}{2\rho} \\ 0 & 0 & \frac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda + 2\mu}{\rho} & 0 \\ -\frac{\mu\beta}{P_r\rho} & \frac{\beta^2}{4}\frac{\mu}{P_r\rho} - \frac{\mu}{2\rho} & 0 & 0 & \frac{\beta^2 u}{4P_r\rho} + \frac{\mu}{2\rho} \end{bmatrix},$$

$$S_{\mathrm{H}}^{-1} E_{xy} S_{\mathrm{H}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda + \mu}{\sqrt{2}\rho} & 0 & 0 \\ 0 & \frac{\lambda + \mu}{\sqrt{2}\rho} & 0 & 0 & -\frac{\lambda + \mu}{\sqrt{2}\rho} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\lambda + \mu}{\sqrt{2}\rho} & 0 & 0 \end{bmatrix},$$

$$S_H^{-1}E_{yz}S_H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\lambda+\mu}{\rho} & 0 \\ 0 & 0 & \dfrac{\lambda+\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_H^{-1}E_{zx}S_H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\lambda+\mu}{\sqrt{2}\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\lambda+\mu}{\sqrt{2}\rho} & 0 & 0 & -\dfrac{\lambda+\mu}{\sqrt{2}\rho} \\ 0 & 0 & 0 & -\dfrac{\lambda+\mu}{\sqrt{2}\rho} & 0 \end{bmatrix}.$$

It turns out, however, that this form of the symmatrized coefficients is not suitable for the stability analysis. This point will become evident in Section IV. We therefore sought another similarity transformation. The construction of this second symmetrizer is based on the observation that the coefficient matrices C, D and K commute and are, therefore, simultaneously diagonalizable. We designate this similarity transformation by S_P to indicate that it is related to the parabolic part of the equation.

$$S_P = \begin{bmatrix} \dfrac{\sqrt{\gamma p}}{c} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \dfrac{\rho c}{\sqrt{\gamma}} & 0 & 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\rho c \end{bmatrix},$$

$$S_P^{-1} = \begin{bmatrix} \dfrac{c}{\sqrt{\gamma p}} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\dfrac{c}{\sqrt{\gamma}\sqrt{\gamma-1}} & 0 & 0 & 0 & \sqrt{\dfrac{\gamma}{\gamma-1}}\dfrac{1}{\rho c} \end{bmatrix}. \tag{2.7}$$

These symmetrized matrix coefficients are simpler than those produced by the first symmetrizer, and are given by

$$S_P^{-1} A S_P = \begin{bmatrix} u & \dfrac{c}{\sqrt{\gamma}} & 0 & 0 & 0 \\ \dfrac{c}{\sqrt{\gamma}} & u & 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}c \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}c & 0 & 0 & u \end{bmatrix},$$

$$S_P^{-1} B S_P = \begin{bmatrix} v & 0 & \dfrac{c}{\sqrt{\gamma}} & 0 & 0 \\ 0 & v & 0 & 0 & 0 \\ \dfrac{c}{\sqrt{\gamma}} & 0 & v & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}c \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}c & 0 & v \end{bmatrix},$$

$$S_P^{-1} J S_P = \begin{bmatrix} w & 0 & 0 & \frac{c}{\sqrt{\gamma}} & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ \frac{c}{\sqrt{\gamma}} & 0 & 0 & w & \sqrt{\frac{\gamma-1}{\gamma}}c \\ 0 & 0 & 0 & \sqrt{\frac{\gamma-1}{\gamma}}c & w \end{bmatrix},$$

$$S_P^{-1} C S_P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda+2\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu}{\rho} & 0 \\ 0 & 0 & 0 & 0 & \frac{\gamma\mu}{P_r\rho} \end{bmatrix},$$

$$S_P^{-1} D S_P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda+2\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu}{\rho} & 0 \\ 0 & 0 & 0 & 0 & \frac{\gamma\mu}{P_r\rho} \end{bmatrix},$$

$$S_P^{-1} K S_P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda+2\mu}{\rho} & 0 \\ 0 & 0 & 0 & 0 & \frac{\gamma\mu}{P_r\rho} \end{bmatrix},$$

$$S_P^{-1}E_{xy}S_P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 \\ 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_P^{-1}E_{yz}S_P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 \\ 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_P^{-1}E_{zx}S_P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.8}$$

Notice that the coefficients of the mixed derivatives remain unchanged under the similarity transformation, being symmetric to being with.

III. The New Splitting

If one is interested in time splitting the Navier–Stokes equations, there are several ways of doing it. MacCormack and Baldwin [4], for example, use the two-dimensional version of (2.2) to indicate how to split the two-dimensional conservation form of the Navier–Stokes equations:

$$V^{n+2} = (L_2 L_1 L_1 L_2)\, V^n, \tag{3.1}$$

where V^n is the finite difference vector approximation to $\mathbf{U}(n\Delta t)$ and L_1 is the finite difference operator mapping $\mathbf{W}(n\Delta t)$ into $\mathbf{W}((n+1)\Delta t)$ using the equations $(\partial \mathbf{W}/\partial t) + (\partial \mathbf{F}/\partial x) = 0$. Similarly L_2 maps $\mathbf{W}(n\Delta t)$ into $\mathbf{W}((n+1)\Delta t)$ simulating $(\partial \mathbf{W}/\partial t) + (\partial \mathbf{G}/\partial y) = 0$. This method means in effect that in the two-dimensional version of Eq. (2.3) the matrix coefficient of the mixed derivative term, E_{xy} has to be split in two:

$$[E_{xy_1}]_{2D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\lambda}{\rho} & 0 \\ 0 & \dfrac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad [E_{xy_2}]_{2D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\mu}{\rho} & 0 \\ 0 & \dfrac{\lambda}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that because $\lambda < 0$ for real fluids, these matrices have imaginary eigenvalues and hence cannot be symmetrized at all. Consequently, as may be easily verified, $\|L_1\|$ and $\|L_2\|$ are greater than unity, which means that this kind of split is not "classical" in the sense of Strang [5].

We now propose a new method of splitting. We first rewrite the conservation-form equations in the suggestive way

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}(\mathbf{F}_{\mathrm{H}} + \mathbf{F}_{\mathrm{P}} + \mathbf{F}_{\mathrm{M}}) + \frac{\partial}{\partial y}(\mathbf{G}_{\mathrm{H}} + \mathbf{G}_{\mathrm{P}} + \mathbf{G}_{\mathrm{M}}) + \frac{\partial}{\partial z}(\mathbf{H}_{\mathrm{H}} + \mathbf{H}_{\mathrm{P}} + \mathbf{H}_{\mathrm{M}}) = 0, \tag{3.2}$$

where the subscripts (H, P, M) denote, respectively, the appropriate hyperbolic, parabolic and mixed derivative parts of the original flux vectors **F**, **G** and **H**. The new fluxes are given by

$$\mathbf{F}_{\mathrm{H}} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ (e+p)\, u \end{bmatrix}, \qquad \mathbf{F}_{\mathrm{P}} = \begin{bmatrix} 0 \\ -(\lambda + 2\mu)\dfrac{\partial u}{\partial x} \\ -\mu \dfrac{\partial v}{\partial x} \\ -\mu \dfrac{\partial w}{\partial x} \\ -(\lambda + 2\mu)\, u \dfrac{\partial u}{\partial x} - \mu v \dfrac{\partial v}{\partial x} - \mu w \dfrac{\partial w}{\partial x} - k \dfrac{\partial T}{\partial x} \end{bmatrix}, \tag{3.3}$$

$$\mathbf{F}_{\mathrm{M}} = \begin{bmatrix} 0 \\ -\lambda \left(\dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) \\ -\mu \dfrac{\partial u}{\partial y} \\ -\mu \dfrac{\partial u}{\partial z} \\ -\lambda \left(\dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) u - \mu v \dfrac{\partial u}{\partial y} - \mu w \dfrac{\partial u}{\partial z} \end{bmatrix};$$

$$\mathbf{G}_{\mathrm{H}} = \begin{bmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ \rho v w \\ (e + p) v \end{bmatrix}, \qquad \mathbf{G}_{\mathrm{P}} = \begin{bmatrix} 0 \\ -\mu \dfrac{\partial u}{\partial y} \\ -(\lambda + 2\mu) \dfrac{\partial v}{\partial y} \\ -\mu \dfrac{\partial w}{\partial y} \\ -(\lambda + 2\mu) v \dfrac{\partial v}{\partial y} - \mu w \dfrac{\partial w}{\partial y} - \mu u \dfrac{\partial u}{\partial y} - k \dfrac{\partial T}{\partial y} \end{bmatrix}, \tag{3.4}$$

$$\mathbf{G}_{\mathrm{M}} = \begin{bmatrix} 0 \\ -\mu \dfrac{\partial v}{\partial x} \\ -\lambda \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z}\right) \\ -\mu \dfrac{\partial v}{\partial z} \\ -\lambda \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z}\right) v - \mu w \dfrac{\partial v}{\partial z} - \mu u \dfrac{\partial v}{\partial x} \end{bmatrix};$$

$$\mathbf{H}_{\mathrm{H}} = \begin{bmatrix} \rho w \\ \rho w u \\ \rho w v \\ \rho w^2 + p \\ (e + p) w \end{bmatrix}, \qquad \mathbf{H}_{\mathrm{P}} = \begin{bmatrix} 0 \\ -\mu \dfrac{\partial u}{\partial z} \\ -\mu \dfrac{\partial v}{\partial z} \\ -(\lambda + 2\mu) \dfrac{\partial w}{\partial z} \\ -(\lambda + 2\mu) w \dfrac{\partial w}{\partial z} - \mu u \dfrac{\partial u}{\partial z} - \mu v \dfrac{\partial v}{\partial z} - k \dfrac{\partial T}{\partial z} \end{bmatrix}, \tag{3.5}$$

$$\mathbf{H}_{\mathrm{M}} = \begin{bmatrix} 0 \\ -\mu \dfrac{\partial w}{\partial x} \\ -\mu \dfrac{\partial w}{\partial y} \\ -\lambda \left(\dfrac{\partial v}{\partial y} + \dfrac{\partial u}{\partial x} \right) \\ -\lambda \left(\dfrac{\partial v}{\partial y} + \dfrac{\partial u}{\partial x} \right) w - \mu u \dfrac{\partial w}{\partial x} - \mu v \dfrac{\partial w}{\partial y} \end{bmatrix}.$$

This leads to the following time splitting:

Let L_x be a finite difference scheme that solves $(\partial \mathbf{U}/\partial t) + (\partial \mathbf{F}_{\mathrm{H}}/\partial x) = 0$.

Let L_{xx} be a finite difference scheme that solves $(\partial \mathbf{U}/\partial t) + (\partial \mathbf{F}_{\mathrm{P}}/\partial x) = 0$.

Let L_y be a finite difference scheme that solves $(\partial \mathbf{U}/\partial t) + (\partial \mathbf{G}_{\mathrm{H}}/\partial y) = 0$.

Let L_{yy} be a finite difference scheme that solve $(\partial \mathbf{U}/\partial t) + (\partial \mathbf{G}_{\mathrm{P}}/\partial y) = 0$.

Let L_z be a finite difference scheme that solves $(\partial \mathbf{U}/\partial t) + (\partial \mathbf{H}_{\mathrm{H}}/\partial z) = 0$.

Let L_{zz} be a finite difference scheme that solves $(\partial \mathbf{U}/\partial t) + (\partial \mathbf{H}_{\mathrm{P}}/\partial z) = 0$.

Let L_{xyz} be a finite difference scheme that solves $(\partial \mathbf{U}/\partial t) + (\partial \mathbf{F}_{\mathrm{M}}/\partial x) + (\partial \mathbf{G}_{\mathrm{M}}/\partial y) + (\partial \mathbf{H}_{\mathrm{M}}/\partial z) = 0$.

Then the new algorithm is constructed as

$$\begin{aligned} U^{n+2} = \{ & [L_x(\Delta t_x)\, L_y^{\tau_y}(\Delta t_y)\, L_z^{\tau_z}(\Delta t_z)\, L_{xyz}(\Delta t_{xyz})\, L_{xx}^{\sigma}(\Delta t_{xx})\, L_{yy}^{\sigma S^2}(\Delta t_{yy})\, L_{zz}^{\sigma Q^2}(\Delta t_{zz})] \\ & \cdot [L_{zz}^{\sigma Q^2}(\Delta t_{zz})\, L_{yy}^{\sigma S^2}(\Delta t_{yy})\, L_{xx}^{\sigma}(\Delta t_{xx})\, L_{xyz}(\Delta t_{xyz})\, L_z^{\tau_z}(\Delta t_z)\, L_y^{\tau_y}(\Delta t_y)\, L_x(\Delta t_x)]\}\, U^n, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \tau_y = [\Delta t_x/\Delta t_y], \qquad \tau_z = [\Delta t_x/\Delta t_z], \qquad \sigma = [\Delta t_x/\Delta t_{xx}], \\ S^2 = [\Delta t_{xx}/\Delta t_{yy}], \qquad Q^2 = [\Delta t_{xx}/\Delta t_{zz}], \end{aligned} \tag{3.7}$$

and $[\cdot]$ designates the nearest greater integer, unless the quantity in question happens to be an integer and then $[\cdot]$ takes this integer value. If we deal with second order accurate schemes for each of the operators then, within linear stability for each of them, S, Q, σ, τ_y and τ_z take the following meanings:

$S = [(r(D)(\Delta x)^2/r(C)(\Delta y)^2)^{1/2}]$ is related to the mesh stretching, assuming $\Delta x \geqslant \Delta y$,

$Q = [(r(K)(\Delta x)^2/r(C)(\Delta z)^2]^{1/2}]$ is related to the mesh stretching in the other direction $(\Delta x \geqslant \Delta z)$,

$\sigma = [2r(C)/r(A)(\Delta x)]$ is twice the reciprocal of the cell Reynolds number,

$\tau_y = [r(B)(\Delta x)/r(A)(\Delta y)]$,

$\tau_z = [r(J)(\Delta x)/r(A)(\Delta z)]$, where $r(A)$, $r(B)$, $r(J)$, $r(C)$, $r(D)$, $r(K)$ are, respectively, the spectral radii of the coefficient matrices A, B, J, C, D and K defined in Section II. The above relations determine (from the point of view of accuracy) Δt_y, Δt_z, Δt_{xx}, Δt_{yy} and Δt_{zz} as functions of Δt_x. The various subscripts of Δt indicate the corresponding finite difference operator and thus, for example, Δt_x is given by the stability condition of L_x, i.e., $\Delta t_x \leqslant \Delta x/r(A)$ (at least for all centered second order one-dimensional schemes). Notice that (3.6) implies that Δt_{xyz} is taken equal to Δt_x. This very surprising result is established in Section IV.

We now present an explicit example of an implementation of the above procedure based on the MacCormack scheme. We start with the hyperbolic split operator in the x-direction

$$U^*_{j,k,l} = U^n_{j,k,l} - \frac{\Delta t_x}{\Delta x}\,(F^n_{\mathrm{H}j+1,k,l} - F^n_{\mathrm{H}j,k,l}),$$

$$U^{**}_{j,k,l} = U^*_{j,k,l} - \frac{\Delta t_x}{\Delta x}\,(F^*_{\mathrm{H}j,k,l} - F^*_{\mathrm{H}j-1,k,l}),$$

$$U^{n+1}_{j,k,l} = \tfrac{1}{2}(U^n_{j,k,l} + U^{**}_{j,k,l}) \equiv L_x(\Delta t_x)\, U^n_{j,k,l}. \tag{3.8}$$

Here we used the standard notation for finite difference schemes. Next consider the parabolic split operator in the x-direction

$$U^*_{j,k,l} = U^n_{j,k,l} - \frac{\Delta t_{xx}}{\Delta x}\,(F^n_{\mathrm{P}j+1,k,l} - F^n_{\mathrm{P}j,k,l}),$$

where the x-derivative terms appearing in F^n_{P} are expressed by *backward* differencing;

$$U^{**}_{j,k,l} = U^*_{j,k,l} - \frac{\Delta t_{xx}}{\Delta x}\,(F^*_{\mathrm{P}j,k,l} - F^*_{\mathrm{P}j-1,k,l}),$$

here, however, the x-derivatives inside F^*_{P} are expressed by *forward differencing*;

$$U^{n+1}_{j,k,l} = \tfrac{1}{2}(U^n_{j,k,l} + U^{**}_{j,k,l}) \equiv L_{xx}(\Delta t_{xx})\, U^n_{j,k,l}. \tag{3.9}$$

Similarly, for the hyperbolic split operators in the y and z directions we have

$$U^*_{j,k,l} = U^n_{j,k,l} - \frac{\Delta t_y}{\Delta y}\,(G^n_{\mathrm{H}j,k+1,l} - G^n_{\mathrm{H}j,k,l}),$$

$$U^{**}_{j,k,l} = U^*_{j,k,l} - \frac{\Delta t_y}{\Delta y}\,(G^*_{\mathrm{H}j,k,l} - G^*_{\mathrm{H}j,k-1,l}),$$

$$U^{n+1}_{j,k,l} = \tfrac{1}{2}(U^n_{j,k,l} + U^{**}_{j,k,l}) \equiv L_y(\Delta t_y)\, U^n_{j,k,l}. \tag{3.10}$$

and

$$U^*_{j,k,l} = U^n_{j,k,l} - \frac{\Delta t_z}{\Delta z}\left(H^n_{\mathrm{H}j,k,l+1} - H^n_{\mathrm{H}j,k,l}\right),$$

$$U^{**}_{j,k,l} = U^*_{j,k,l} - \frac{\Delta t_z}{\Delta z}\left(H^*_{\mathrm{H}j,k,l} - H^*_{\mathrm{H}j,k,l-1}\right),$$

$$U^{n+1}_{j,k,l} = \tfrac{1}{2}\left(U^n_{j,k,l} + U^{**}_{j,k,l}\right) \equiv L_z(\Delta t_z)\, U^n_{j,k,l}. \tag{3.11}$$

The parabolic split operators in the y and z directions (with the order of backward and forward differencing being the same as in the x-split parabolic operator, see (3.9)) are given by

$$U^*_{j,k,l} = U^n_{j,k,l} - \frac{\Delta t_{yy}}{\Delta y}\left(G^n_{\mathrm{P}j,k+1,l} - G^n_{\mathrm{P}j,k,l}\right),$$

$$U^{**}_{j,k,l} = U^*_{j,k,l} - \frac{\Delta t_{yy}}{\Delta y}\left(G^*_{\mathrm{P}j,k,l} - G^*_{\mathrm{P}j,k-1,l}\right),$$

$$U^{n+1}_{j,k,l} = \tfrac{1}{2}\left(U^n_{j,k,l} + U^{**}_{j,k,l}\right) \equiv L_{yy}(\Delta t_{yy})\, U^n \tag{3.12}$$

and

$$U^*_{j,k,l} = U^n_{j,k,l} - \frac{\Delta t_{zz}}{\Delta z}\left(H^*_{\mathrm{P}j,k,l+1} - H^n_{\mathrm{P}j,k,l}\right),$$

$$U^{**}_{j,k,l} = U^*_{j,k,l} - \frac{\Delta t_{zz}}{\Delta z}\left(H^n_{\mathrm{P}j,k,l} - H^n_{\mathrm{P}j,k,l-1}\right),$$

$$U^{n+1}_{j,k,l} = \tfrac{1}{2}\left(U^n_{j,k,l} + U^{**}_{j,k,l}\right) \equiv L_{zz}(\Delta t_{zz})\, U^n. \tag{3.13}$$

We now come to the definition of L_{xyz}. Since the MacCormack differencing method cannot be applied in a straightforward manner to the case of mure mixed derivatives, we propose a differencing algorithm which is MacCormack-like in its structure:

$$U^*_{j,k,l} = U^n_{j,k,l} - \frac{\Delta t_{xyz}}{2\Delta x}\left(F^n_{\mathrm{M}j+1,k,l} - F^n_{\mathrm{M}j-1,k,l}\right)$$
$$- \frac{\Delta t_{xyz}}{2\Delta y}\left(G^n_{\mathrm{M}j,k+1,l} - G^n_{\mathrm{M}j,k-1,l}\right) - \frac{\Delta t_{xyz}}{2\Delta z}\left(H^n_{\mathrm{M}j,k,l+1} - H^n_{\mathrm{M}j,k,l-1}\right),$$

$$U^{**}_{j,k,l} = U^*_{j,k,l} - \frac{\Delta t_{xyz}}{2\Delta x}\left(F^*_{\mathrm{M}j+1,k,l} - F^*_{\mathrm{M}j-1,k,l}\right)$$
$$- \frac{\Delta t_{xyz}}{2\Delta y}\left(G^*_{\mathrm{M}j,k+1,l} - G^*_{\mathrm{M}j,k-1,l}\right) - \frac{\Delta t_{xyz}}{2\Delta z}\left(H^*_{\mathrm{M}j,k,l+1} - H^*_{\mathrm{M}j,k,l-1}\right),$$

$$U^{n+1}_{j,k,l} = \tfrac{1}{2}\left(U^n_{j,k,l} + U^{**}_{j,k,l}\right) \equiv L_{xyz}(\Delta t_{xyz})\, U^n_{j,k,l}, \tag{3.14}$$

where all the derivatives appearing in F^n_{M}, F^*_{M}, G^n_{M}, G^*_{M}, H^n_{M}, H^*_{M} are expressed by *central* finite differencing.

We are now in a position to construct algorithm (3.6) unambiguously for the MacCormack scheme and in a similar vain construct it for other finite difference schemes.

It may be shown, using the results in [9], that when we use (3.8) to (3.14) to implement algorithm (3.6), the overall scheme is second order accurate in time and space. The numerical calculations described in Section V verify this conclusion.

IV. Stability Analysis

We now examine the (linear) stability of second order accurate finite difference approximations to a scalar parabolic partial differential equation which models the Navier–Stokes system. We then show that the stability criteria which follow from the study of the scalar case generalize appropriately to the full (linearized) Navier–Stokes equations.

Consider the model scalar equation with constant coefficients

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + j\frac{\partial u}{\partial z} + c\frac{\partial^2 u}{\partial x^2} + d\frac{\partial^2 u}{\partial y^2} + k\frac{\partial^2 u}{\partial z^2} + e_{xy}\frac{\partial^2 u}{\partial x\,\partial y} + e_{yz}\frac{\partial^2 u}{\partial y\,\partial z} + e_{zx}\frac{\partial^2 u}{\partial z\,\partial x}. \tag{4.1}$$

The requirement for parabolicity is that the quadratic form

$$c\omega_1^2 + d\omega_2^2 + k\omega_3^2 + e_{xy}\omega_1\omega_2 + e_{yz}\omega_2\omega_3 + e_{zx}\omega_3\omega_1 \geqslant 0, \tag{4.2}$$

$\forall -\infty < \omega_1, \omega_2, \omega_3 < \infty$. Necessary (but definitely not sufficient) conditions for meeting the definition of parabolicity (4.2) (besides having c, d and k positive) are

$$e_{xy}\omega_1\omega_2 \leqslant c\omega_1^2 + d\omega_2^2 \Rightarrow e_{xy}^2 \leqslant 4cd, \tag{4.3a}$$

$$e_{yz}\omega_2\omega_3 \leqslant d\omega_2^2 + k\omega_3^2 \Rightarrow e_{yz}^2 \leqslant 4dk, \tag{4.3b}$$

$$e_{zx}\omega_3\omega_1 \leqslant k\omega_3^2 + c\omega_1^2 \Rightarrow e_{zx}^2 \Rightarrow 4kc. \tag{4.3c}$$

We note in passing that in the two-dimensional case ($e_{yz} = e_{zx} = 0$) the necessary condition $e_{xy} \leqslant 4cd$ is also sufficient.

Adding both sides of inequalities (4.3) we get

$$c\omega_1^2 + d\omega_2^2 + k\omega_3^2 - \tfrac{1}{2}(e_{xy}\omega_1\omega_2 + e_{yz}\omega_2\omega_3 + e_{zx}\omega_3\omega_1) \geqslant 0. \tag{4.4}$$

Comparing (4.4) with (4.2) we finally get a result which is used later,

$$c\omega_1^2 + d\omega_2^2 + k\omega_3^2 - \tfrac{1}{2}\,|e_{xy}\omega_1\omega_2 + e_{yz}\omega_2\omega_3 + e_{zx}\omega_3\omega_1| \geqslant 0. \tag{4.5}$$

It is well known that the amplification factor for each split operator is the same for all three point centered finite difference schemes and is, in fact, the appropriate Lax–Wendroff one [10]. If we designate by J_x the amplification factor corresponding to L_x, and similarly for the other split operators, then we have

$$J_x = 1 + 2i\lambda_x \alpha \sqrt{1-\alpha^2} - 2\lambda_x^2 \alpha^2, \tag{4.6a}$$

$$J_y = 1 + 2i\lambda_y \beta \sqrt{1-\beta^2} - 2\lambda_y^2 \beta^2, \tag{4.6b}$$

$$J_z = 1 + 2i\lambda_z \nu \sqrt{1-\nu^2} - 2\lambda_z^2 \nu^2, \tag{4.6c}$$

$$J_{xx} = 1 - 4\lambda_{xx}\alpha^2 + 8\lambda_{xx}^2 \alpha^4, \tag{4.6d}$$

$$J_{yy} = 1 - 4\lambda_{yy}\beta^2 + 8\lambda_{yy}^2 \beta^4, \tag{4.6e}$$

$$J_{zz} = 1 - 4\lambda_{zz}\nu^2 + 8\lambda_{zz}^2 \nu^4, \tag{4.6f}$$

$$\begin{aligned} J_{xyz} = 1 &- 4(\lambda_{xy}\alpha\beta\sqrt{1-\alpha^2}\sqrt{1-\beta^2} + \lambda_{yz}\beta\nu\sqrt{1-\beta^2}\sqrt{1-\nu^2} \\ &+ \lambda_{zx}\nu\alpha\sqrt{1-\alpha^2}\sqrt{1-\nu^2}) \\ &+ 8(\lambda_{xy}\alpha\beta\sqrt{1-\alpha^2}\sqrt{1-\beta^2} + \lambda_{yz}\beta\nu\sqrt{1-\beta^2}\sqrt{1-\nu^2} \\ &+ \lambda_{zx}\nu\alpha\sqrt{1-\alpha^2}\sqrt{1-\nu^2})^2, \end{aligned} \tag{4.6g}$$

where $\alpha = \sin(\xi/2)$, $\beta = \sin(\eta/2)$, $\nu = \sin(\zeta/2)$; ξ, η and ζ being the dual Fourier variables of the space coordinates x, y, z. Also

$$\begin{array}{lll} \lambda_x = a\,\Delta t_x/\Delta x, & \lambda_y = b\,\Delta t_y/\Delta y, & \lambda_z = j\,\Delta t_z/\Delta z, \\ \lambda_{xx} = c\,\Delta t_{xx}/(\Delta x)^2, & \lambda_{yy} = d\,\Delta t_{yy}/(\Delta y)^2, & \lambda_{zz} = k\,\Delta t_{zz}/(\Delta z)^2, \\ \lambda_{xy} = e_{xy}\,\Delta t_{xyz}/\Delta x\,\Delta y, & \lambda_{yz} = e_{yz}\,\Delta t_{xyz}/\Delta y\,\Delta z, & \lambda_{zx} = e_{zx}\,\Delta t_{xyz}/\Delta z\,\Delta x, \end{array}$$

and the various Δt's are the ones belonging to the split operators indicated by the subscripts. Note that while the absolute values of J_x, J_y, J_z, J_{xx}, J_{yy}, and J_{zz} are bounded by unity under their respective one-dimensional stability conditions

$$\lambda_x \leqslant 1, \quad \lambda_y \leqslant 1, \quad \lambda_z \leqslant 1; \qquad \lambda_{xx} \leqslant \tfrac{1}{2}, \quad \lambda_{yy} \leqslant \tfrac{1}{2}, \quad \lambda_{zz} \leqslant \tfrac{1}{2}. \tag{4.7}$$

J_{xyz} is not bounded by 1 and hence, by itself, L_{xyz} is unconditionally unstable. The amplification factor for the whole scheme (3.6) is given by

$$M = (J_x J_{xyz} J_y^{\tau_y} J_z^{\tau_z} J_{xx}^{\sigma} J_{yy}^{\sigma S^2} J_{zz}^{\sigma Q^2})^2.$$

We show that even though $\forall\, \lambda_{xy}, \lambda_{yz}, \lambda_{zx}$, $|J_{xyz}| \geqslant 1$ (for some α, β, ν) M is "stable," i.e., $|M| \leqslant 1$, under conditions (4.7) for λ_x, λ_y, λ_z, λ_{xx}, λ_{yy}, and λ_{zz} provided that $\Delta t_{xyz} \leqslant \Delta t_x$. Since under the specified conditions, $|J_x| \leqslant 1$, $|J_y| \leqslant 1$, $|J_z| \leqslant 1$, it is sufficient to investigate the quantity

$$\tilde{M} = |J_{xyz}| \cdot |J_{xx} J_{yy}^{S^2} J_{zz}^{Q^2}|^{\sigma}. \tag{4.8}$$

Now J_{xyz} is of the form $J_{xyz} = 1 - n + (n^2/2)$, where

$$n = 4(\lambda_{xy}\alpha\beta\sqrt{1-\alpha^2}\sqrt{1-\beta^2} + \lambda_{yz}\beta\nu\sqrt{1-\beta^2}\sqrt{1-\nu^2} + \lambda_{zx}\nu\alpha\sqrt{1-\nu^2}\sqrt{1-\alpha^2}). \tag{4.9}$$

Notice that n may be either positive or negative, and we carry the stability investigation for either case separately. Consider first the case of $n \leqslant 0$, then

$$|J_{xyz}| = 1 - n + (n^2/2) \leqslant e^{-n}. \tag{4.10}$$

Next we evaluate $|J_{xyz}|$ for $n \geqslant 0$ (note that $J_{xyz} \geqslant 0\ \forall n$),

$$|J_{xyz}| = 1 - n + (n^2/2) \leqslant e^{n/2}. \tag{4.11}$$

Referring to (4.6) we have

$$\begin{aligned} J_{xx} &= 1 - h \leqslant e^{-h}, &&\text{with} \quad 0 \leqslant h = 4\lambda_{xx}\alpha^2(1 - 2\lambda_{xx}\alpha^2) \leqslant 4\lambda_{xx}\alpha^2(1-\alpha^2) \leqslant \tfrac{1}{2}, \\ J_{yy} &= 1 - l \leqslant e^{-l}, &&\text{with} \quad 0 \leqslant l = 4\lambda_{yy}\beta^2(1 - 2\lambda_{yy}\beta^2) \leqslant 4\lambda_{yy}\beta^2(1-\beta^2) \leqslant \tfrac{1}{2}, \\ J_{zz} &= 1 - m \leqslant e^{-m}, &&\text{with} \quad 0 \leqslant m = 4\lambda_{zz}\nu^2(1 - 2\lambda_{zz}\nu^2) \leqslant 4\lambda_{zz}\nu^2(1-\nu^2) \leqslant \tfrac{1}{2}. \end{aligned}$$

Therefore

$$J_{xx}^{\sigma} J_{yy}^{\sigma S^2} J_{zz}^{\sigma Q^2} \leqslant \exp\{-4\sigma[\lambda_{xx}\alpha^2(1-\alpha^2) + \lambda_{yy}\beta^2(1-\beta^2)\,S^2 + \lambda_{zz}\nu^2(1-\nu^2)\,Q^2]\}.$$

Using the definitions of λ_{xx}, λ_{yy}, λ_{zz}, σ, S^2 and Q^2 we have

$$|J_{xx} J_{yy}^{S^2} J_{zz}^{Q^2}|^{\sigma} \leqslant \exp\{-c\omega_1^2 - d\omega_2^2 - k\omega_3^2\}, \tag{4.12}$$

where we have defined

$$\begin{aligned} \omega_1 &= 2\alpha\sqrt{1-\alpha^2}\sqrt{\Delta t_x}/\Delta x, \\ \omega_2 &= 2\beta\sqrt{1-\beta^2}\sqrt{\Delta t_x}/\Delta y, \\ \omega_3 &= 2\nu\sqrt{1-\nu^2}\sqrt{\Delta t_x}/\Delta z. \end{aligned} \tag{4.13}$$

We are now ready to consider the two cases of negative and positive n.

(i) *Case* $n \leqslant 0$. Combining (4.10) and (4.12) and using definitions (4.9) and (4.13) we have

$$\tilde{M} \leqslant \exp\{-[(\Delta t_{xyz}/\Delta t_x)(e_{xy}\omega_1\omega_2 + e_{yz}\omega_2\omega_3 + e_{zx}\omega_3\omega_1) + c\omega_1^2 + d\omega_2^2 + k\omega_3^2]\}. \tag{4.14}$$

By referring to the definition of parabolicity (4.2), we get

$$\tilde{M} \leqslant 1, \tag{4.15}$$

provided that $\Delta t_{xyz} \leqslant \Delta t_x$ as claimed above.

(ii) *Case* $n \geqslant 0$. Combining (4.11) and (4.12) and using the same definitions we have

$$\tilde{M} \leqslant \exp\{-[-\tfrac{1}{2}(\Delta t_{xyz}/\Delta t_x)(e_{xy}\omega_1\omega_2 + e_{yz}\omega_2\omega_3 + e_{zx}\omega_3\omega_1) + c\omega_1^2 + d\omega_2^2 + k\omega_3^2]\}.$$

By referring to (4.5), we again get

$$\tilde{M} \leqslant 1,$$

with the same restriction, i.e., $\Delta t_{xtz} \leqslant \Delta t_x$.

Thus the scalar case analysis is completed and we have established the (linear) stability of the model equation (4.1) when solved numerically using algorithm (3.6) with $\Delta t_{xyz} = \Delta t_x$ and conditions (4.7). It should be noted in passing that in the two-dimensional case ($j = k = e_{yz} = e_{zx} = 0$) the stability analysis goes over much more readily because the necessary condition for parabolicity ($e_{xy}^2 \leqslant 4cd$) is also sufficient.

It remains to show that the stability of algorithm (3.6) applied to the Navier–Stokes equations follows from the above scalar analysis. Consider the L_2 norm of (3.6)

$$\|U^{n+2}\| \leqslant \|L_x L_y^{\tau_y} L_z^{\tau_z} L_{xx}^{\sigma} L_{yy}^{\sigma S^2} L_{zz}^{\sigma Q^2} L_{xyz}^2 L_{zz}^{\sigma Q^2} L_{yy}^{\sigma S^2} L_{xx}^{\sigma} L_z^{\tau_z} L_y^{\tau_y} L_x\| \cdot \|U^n\|. \tag{4.16}$$

Now, after symmetrization the norms of the hyperbolic operators L_x, L_y, L_z are less than (or equal to) unity under conditions (4.7), and thus

$$\begin{aligned}\|U^{n+2}\| &\leqslant \|L_{xx}^{\sigma} L_{yy}^{\sigma S^2} L_{zz}^{\sigma Q^2} L_{xyz}^2 L_{zz}^{\sigma Q^2} L_{yy}^{\sigma S^2} L_{xx}^{\sigma}\| \cdot \|U^n\| \\ &= \|L_{xx}^{\sigma} L_{yy}^{\sigma S^2} L_{zz}^{\sigma Q^2} L_{xyz}\|^2 \cdot \|U^n\|,\end{aligned}$$

where the last equality is due to the parabolicity and symmetry of the indicated split operators. The requirement for stability is then

$$\|L_{xx}^{\sigma} L_{yy}^{\sigma S^2} L_{zz}^{\sigma Q^2} L_{xyz}\| \leqslant 1. \tag{4.17}$$

The corresponding amplification matrix is found from (4.6d)–(4.6g) by replacing the various scalar coefficients in the λ's by the appropriate corresponding matrix coefficients

$$N = J_{xx}^{\sigma} J_{yy}^{\sigma S^2} J_{zz}^{\sigma Q^2} J_{xyz},$$

where, as indicated above,

$$\lambda_{xx} = (\Delta t_{xx}/(\Delta x)^2)\, S_P^{-1} C S_P, \qquad \lambda_{yy} = (\Delta t_{yy}/(\Delta y)^2)\, S_P^{-1} D S_P,$$
$$\lambda_{zz} = (\Delta t_{zz}/(\Delta z)^2)\, S_P^{-1} K S_P,$$
$$\lambda_{xy} = (\Delta t_{xyz}/\Delta x\, \Delta y)\, S_P^{-1} E_{xy} S_P, \qquad \lambda_{yz} = (\Delta t_{xyz}/\Delta y\, \Delta z)\, S_P^{-1} E_{yz} S_P,$$
$$\lambda_{zx} = (\Delta t_{xyz}/(\Delta z\, \Delta x))\, S_P^{-1} E_{zx} S_P.$$

Notice that the matrices J_{xx}, J_{yy}, J_{zz}, J_{xyz} have the structure

$$J_{xx} = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & \tilde{J}_{xx} & & \end{array}\right], \qquad J_{yy} = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & \tilde{J}_{yy} & & \end{array}\right],$$

$$J_{zz} = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & \tilde{J}_{zz} & & \end{array}\right], \qquad J_{xyz} = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & \tilde{J}_{xyz} & & \end{array}\right],$$

where the lower 4×4 matrices $\tilde{J}_{xx}$, $\tilde{J}_{yy}$, $\tilde{J}_{zz}$, $\tilde{J}_{xyz}$ depend only on the 4×4 lower matrices $\tilde{C}$, $\tilde{D}$, $\tilde{K}$, $\tilde{E}_{xy}$, $\tilde{E}_{yz}$, $\tilde{E}_{zx}$ which appear in $S_P^{-1}CS_P$, $S_P^{-1}DS_P$, $S_P^{-1}KS_P$, $S_P^{-1}E_{xy}S_P$, $S_P^{-1}E_{yz}S_P$ and $S_P^{-1}E_{zx}S_P$. As a consequence the amplification matrix N also has a similar structure.

$$N = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & \tilde{N} = \tilde{J}^{\sigma}_{xx}\tilde{J}^{\sigma S^2}_{yy}\tilde{J}^{\sigma Q^2}_{zz}\tilde{J}_{xyz} & & \end{array}\right]. \tag{4.18}$$

The stability requirement (4.17) is equivalent to $\|N\| \leqslant 1$. It is readily seen from (4.18) that

$$\|N\| = \max\{1; \|\tilde{N}\|\}.$$

If we can show that $\|\tilde{N}\| = \|\tilde{J}^{\sigma}_{xx}\tilde{J}^{\sigma S^2}_{yy}\tilde{J}^{\sigma Q^2}_{zz}\tilde{J}_{xyz}\| \leqslant 1$, then the stability of the algorithm would have been established. We proceed as follows:

$$\|\tilde{N}\| \leqslant \|\tilde{J}_{xyz}\| \cdot \|\tilde{J}_{xx}\|^{\sigma} \cdot \|\tilde{J}_{yy}\|^{\sigma S^2} \cdot \|\tilde{J}_{zz}\|^{\sigma Q^2}.$$

Since $\tilde{J}_{xx}$, $\tilde{J}_{yy}$ and $\tilde{J}_{zz}$ are symmetric we may replace their norms by their respective spectral radii

$$\|\tilde{N}\| \leqslant \|\tilde{J}_{xyz}\| \, r^{\sigma}(\tilde{J}_{xx}) \, r^{\sigma S^2}(\tilde{J}_{yy}) \, r^{\sigma Q^2}(\tilde{J}_{zz}). \tag{4.19}$$

If we identify λ_{xx}, λ_{yy} and λ_{zz} respectively with $(\Delta t_{xx}/(\Delta x)^2)\, C_i$, $(\Delta t_{yy}/(\Delta y)^2)\, D_i$, $(\Delta t_{zz}/(\Delta z)^2)\, K_i$, where C_i, D_i and K_i are any one of the eigenvalues of the indicated coefficient matrices, then by (4.12) we may replace (4.19) by

$$\|\tilde{N}\| \leqslant \|\tilde{J}_{xyz}\| \cdot \exp\{-(C_i\omega_i^2 + D_i\omega_2^2 + K_i\omega_3^2)\}. \tag{4.20}$$

From (4.6g),

$$\|\tilde{J}_{xyz}\| = \|I - \tilde{n} + (\tilde{n}^2/2)\|,$$

where

$$\tilde{n} = (\tilde{E}_{xy}\omega_1\omega_2 + \tilde{E}_{yz}\omega_2\omega_3 + \tilde{E}_{zx}\omega_3\omega_1)(\Delta t_{xyz}/\Delta t_x), \tag{4.21}$$

and now we estimate

$$\|\tilde{J}_{xyz}\| \leqslant 1 + \|\tilde{n}\| + (\|\tilde{n}^2\|/2) = 1 + r(\tilde{n}) + (r^2(\tilde{n})/2) \leqslant e^{r(\tilde{n})}. \tag{4.22}$$

Substituting (4.22) into (4.20) leads to the requirement for stability

$$r(\tilde{n}) \leqslant c_i\omega_1^2 + D_i\omega_2^2 + K_i\omega_3^2. \tag{4.23}$$

In our case, $r(\tilde{E}_{xy}) = r(\tilde{E}_{yz}) = r(\tilde{E}_{zx}) = (\lambda + \mu)/\rho$ and so (4.23) becomes

$$\begin{aligned}(\Delta t_{xyz}/\Delta t_x)((\lambda + \mu)/\rho)(\omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1)\\ \leqslant C_i\omega_1^2 + D_i\omega_2^2 + K_i\omega_3^2,\end{aligned} \tag{4.24}$$

where ω_1, ω_2, ω_3 are now taken to be positive. Since $(\lambda + \mu)/\rho$ is smaller than any of the C_i's, D_i's, and K_i's $((\lambda + 2\mu)/\rho,\ \mu/\rho,\ \gamma\mu/P_r\rho)$, inequality (4.24), with $\Delta t_{xyz} \leqslant \Delta t_x$, is always valid. Thus stability has been established for the Navier–Stokes equations.

V. Numerical Results

In this section we would like to present some numerical experiments with algorithm (3.6) and (A.3.6). We start by considering the scalar two-dimensional equation

$$u_t = u_{xx} + u_{yy} + u_{xy}, \qquad 1 \leqslant x, y \leqslant 2, \tag{5.1}$$

with initial conditions

$$u(x, y, 0) = \cos 2\pi(x - y), \tag{5.2}$$

and periodic boundary conditions

$$\begin{aligned} u(1, y, t) &= u(2, y, t),\\ u(x, 1, t) &= u(x, 2, t).\end{aligned} \tag{5.3}$$

The analytic solution of (5.1)–(5.3) is given by

$$u(x, y, t) = e^{-4\pi^2 t} \cos 2\pi(x - y). \tag{5.4}$$

We attempted to solve numerically Eqs. (5.1)–(5.3) using the algorithm defined by (A.3.6)–(A.3.12). Since (5.1) defines a parabolic equation without any "hyperbolic" term we have to set $\Delta t_x = \Delta t_y = 0$ in (A.3.6). For this same reason we take $\Delta t_{xy} = \Delta t_{xx}$. The other equantities appearing in (A3.6) are taken to be

$$\Delta t_{xy} = \Delta t_{xx}, \qquad \sigma = 1, \qquad S = \Delta x/\Delta y, \qquad \Delta t_{yy} = S^2 \Delta t_{xx}. \tag{5.5}$$

We advanced the solution from $t = 0$ to $t = 0.1$ with different values for Δx and S. The results are summarized in Table I. The first column gives the number of points in the x-direction, the second gives the number of points in the y-direction. The third column gives the value of S. The fourth column consists of the relative L_2 errors, that is, the L_2 norm of the error of the numerical results divided by the L_2 norms of the analytic solution. The fifth column gives the maximum error.

It is clear from Table I that algorithm (A.3.6) is stable. Moreover comparing the L_2 error for a mesh of 30×60 and that of 40×80 one gets the factor $0.98 \times (\frac{4}{3})^2$ which demonstrates the second order accuracy of the algorithm.

The second problem to be considered is the three-dimensional equation

$$u_t = u_{xx} + u_{yy} + u_{zz} + u_{xy} + u_{yz} + u_{xz}, \tag{5.6}$$

with

$$u(x, y, z, 0) = \cos 2\pi(x - y) + \cos 2\pi(x - z) + \cos 2\pi(y - z) + 1,$$

and periodic boundary conditions in the cube $1 \leqslant x, y, z \leqslant 2$.

The solution to (5.6) is given by

$$u(x, y, z, t) = 1 + e^{-4\pi^2 t}[u(x, y, z, 0) - 1]. \tag{5.7}$$

We solve numerically equation (5.6) using algorithm (3.6)–(3.11) with $\Delta t_x = \Delta t_y = \Delta t_z = 0$,

$$\sigma = 1, \quad \Delta t_{xx} = \tfrac{1}{2}(\Delta x)^2, \quad \Delta t_{xyz} = \Delta t_{xx}, \quad \Delta t_{yy} = S^{-2} \Delta t_{xx}, \quad \Delta t_{zz} = Q^{-2} \Delta t_{xx},$$

TABLE I

$\frac{1}{\Delta x}$	$\frac{1}{\Delta y}$	S	L_2 error	max. error
20	20	1	4.8×10^{-2}	1.7×10^{-3}
20	20	2	3×10^{-2}	1×30^{-3}
30	60	2	1.4×10^{-2}	5.1×10^{-4}
40	80	2	8×10^{-3}	2.9×10^{-4}
20	80	4	2.6×10^{-2}	8.6×10^{-4}

TABLE II

$\frac{1}{\Delta x}$	$\frac{1}{\Delta y}$	$\frac{1}{\Delta z}$	S	Q	L_2 error	max. error
10	20	80	2	4	1.8×10^{-3}	6.7×10^{-3}
10	10	10	1	1	2.8×10^{-3}	1.2×10^{-2}
20	20	20	1	1	9.8×10^{-4}	3.8×10^{-3}
30	30	30	1	1	4.6×10^{-4}	1.7×10^{-3}

for different values of x, S, and Q and integrating between $t = 0$ and $t = 0.1$. The results are summarized in Table II. It is evident from Table II that the algorithm is stable. Moreover comparing the L_2 error for the mesh of $20 \times 20 \times 20$ and that for the mesh of $30 \times 30 \times 30$, one obtains a reduction by the factor of $0.95 \times (\frac{3}{2})^2$ which verifies the second order accuracy of (3.6).

VI. Summary

The following results have been presented in this paper:

1. A similarity transformation has been found that symmetrizes simultaneously all nine matrix coefficients of the full three-dimensional Navier–Stokes equations (see Eq. (2.7)).

2. A new time splitting algorithm (see (3.6)) has been constructed which has the following properties:

(i) It is second order accurate in time and has the spatial accuracy of its component operators all of which are one dimensional and either purely hyperbolic or purely parabolic (with the exception of the mixed derivatives operator).

(ii) For any explicit, centered, second order accurate finite difference scheme (such as Lax–Wendroff of MacCormack) the stability of the algorithm has been proven analytically and verified by numerical experiments.

(iii) This stability is achieved with optimal time-steps for all the one-dimensional operators with the surprising result that the mixed derivatives operator may be advanced with a time-step equal to the largest of the others.

3. This algorithm can be cast in conservation form (see Eqs. (3.2)–(3.5) and also (3.8)–(3.14)).

APPENDIX A: The Two-Dimensional Case

A.II. *The Two-Dimensional Navier–Stokes Equations*

The equations corresponding to (2.1) and (2.2) are

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0 \qquad \text{(A.2.1)}$$

with

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + \tau_{xx} \\ \rho uv + \tau_{yx} \\ (e + \tau_{xx})\, u + \tau_{yx} v - k \dfrac{\partial T}{\partial x} \end{bmatrix}, \tag{A.2.2}$$

$$\mathbf{G} = \begin{bmatrix} \rho v \\ \rho vu \\ \rho v^2 + \tau_{yy} \\ (e + \tau_{yy})\, v + \tau_{xy} u - k \dfrac{\partial T}{\partial y} \end{bmatrix}$$

and

$$\tau_{xx} = p - 2\mu \frac{\partial u}{\partial x} - \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),$$

$$\tau_{yy} = p - 2\mu \frac{\partial v}{\partial y} - \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),$$

$$\tau_{xy} = \tau_{yx} = -\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

The non-conservative form of the Navier–Stokes equations is (see Eq. (2.3))

$$\frac{\partial \mathbf{V}}{\partial t} + A \frac{\partial \mathbf{V}}{\partial x} + B \frac{\partial \mathbf{V}}{\partial y} = C \frac{\partial^2 \mathbf{V}}{\partial x^2} + D \frac{\partial^2 \mathbf{V}}{\partial y^2} + E \frac{\partial^2 \mathbf{V}}{\partial x \, \partial y}, \tag{A.2.3}$$

with

$$\bar{V} = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}, \qquad A = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & 1/\rho \\ 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & u \end{bmatrix}, \qquad B = \begin{bmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & 1/\rho \\ 0 & 0 & \gamma p & v \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{\lambda + 2\mu}{\rho} & 0 & 0 \\ 0 & 0 & \dfrac{\mu}{\rho} & 0 \\ -\dfrac{\gamma\mu p}{P_r\rho^2} & 0 & 0 & \dfrac{\gamma u}{P_r\rho} \end{bmatrix}, \tag{A.2.4}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & \dfrac{\lambda + 2\mu}{\rho} & 0 \\ -\dfrac{\gamma\mu p}{P_r\rho^2} & 0 & 0 & \dfrac{\gamma\mu}{P_r\rho} \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\lambda + \mu}{\rho} & 0 \\ 0 & \dfrac{\lambda + \mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The "hyperbolic" symmetrizer, corresponding to Eq. (2.5) is

$$S_H = \begin{bmatrix} \beta\rho & \rho & 0 & \rho \\ 0 & c & 0 & -c \\ 0 & 0 & \sqrt{2}c & 0 \\ 0 & \rho c^2 & 0 & \rho c^2 \end{bmatrix},$$

$$S_H^{-1} = \begin{bmatrix} \dfrac{1}{\beta\rho} & 0 & 0 & -\dfrac{1}{\beta\rho c^2} \\ 0 & \dfrac{1}{2c} & 0 & \dfrac{1}{2\rho c^2} \\ 0 & 0 & \dfrac{1}{\sqrt{2}c} & 0 \\ 0 & -\dfrac{1}{2c} & 0 & \dfrac{1}{2\rho c^2} \end{bmatrix} \tag{A.2.5}$$

with $c = (\gamma p/\rho)^{1/2}$ and $\beta = \sqrt{2(\gamma - 1)}$. The symmetrized matrix coefficients are given by

$$S_{\mathrm{H}}^{-1} A S_{\mathrm{H}} = \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u+c & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u-c \end{bmatrix},$$

$$S_{\mathrm{H}}^{-1} B S_{\mathrm{H}} = \begin{bmatrix} v & 0 & 0 & 0 \\ 0 & v & \dfrac{c}{\sqrt{c}} & 0 \\ 0 & \dfrac{c}{\sqrt{2}} & v & \dfrac{c}{\sqrt{2}} \\ 0 & 0 & \dfrac{c}{\sqrt{2}} & v \end{bmatrix}$$

$$S_{\mathrm{H}}^{-1} C S_{\mathrm{H}} = \begin{bmatrix} \dfrac{\mu}{P_r \rho} & -\dfrac{\beta\mu}{2P_r\rho} & 0 & -\dfrac{\beta\mu}{2P_r\rho} \\ -\dfrac{\beta\mu}{2P_r\rho} & \dfrac{\beta^2\mu}{4P_r\rho} + \dfrac{\lambda+2\mu}{2\rho} & 0 & \dfrac{\beta^2\mu}{4P_r\rho} - \dfrac{\lambda+2\mu}{2\rho} \\ 0 & 0 & \dfrac{\mu}{\rho} & 0 \\ -\dfrac{\beta\mu}{2P_r\rho} & \dfrac{\beta^2\mu}{4P_r\rho} - \dfrac{\lambda+2\mu}{2\rho} & 0 & \dfrac{\beta^2 u}{4P_r\rho} + \dfrac{\lambda+2\mu}{2\rho} \end{bmatrix} \tag{A.2.6}$$

$$S_{\mathrm{H}}^{-1} D S_{\mathrm{H}} = \begin{bmatrix} \dfrac{\mu}{P_r \rho} & -\dfrac{\beta\mu}{2P_r\rho} & 0 & -\dfrac{\beta\mu}{2P_r\rho} \\ -\dfrac{\beta\mu}{2P_r\rho} & \dfrac{\beta^2\mu}{4P_r\rho} + \dfrac{\mu}{2\rho} & 0 & \dfrac{\beta^2\mu}{4P_r\rho} - \dfrac{\mu}{2\rho} \\ 0 & 0 & \dfrac{\lambda+2\mu}{\rho} & 0 \\ -\dfrac{\beta\mu}{2P_r\rho} & \dfrac{\beta^2\mu}{4P_r\rho} - \dfrac{\mu}{2\rho} & 0 & \dfrac{\beta^2 u}{4P_r\rho} + \dfrac{\mu}{2\rho} \end{bmatrix}$$

$$S_H^{-1} E S_H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\lambda+\mu}{\sqrt{2}\rho} & 0 \\ 0 & \dfrac{\lambda+\mu}{\sqrt{2}\rho} & 0 & -\dfrac{\lambda+\mu}{\sqrt{2}\rho} \\ 0 & 0 & -\dfrac{\lambda+\mu}{\sqrt{2}\rho} & 0 \end{bmatrix}.$$

The "parabolic" symmetrizer, corresponding to (2.7) is

$$S_P = \begin{bmatrix} \dfrac{\sqrt{\gamma p}}{c} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{\rho c}{\sqrt{\gamma}} & 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\rho c \end{bmatrix},$$

$$S_P^{-1} = \begin{bmatrix} \dfrac{c}{\sqrt{\gamma p}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\dfrac{c}{\sqrt{\gamma(\gamma-1)}\,\rho} & 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}\dfrac{1}{\rho c} \end{bmatrix}, \tag{A.2.7}$$

and the symmetrized coefficients are then

$$S_P^{-1} A S_P = \begin{bmatrix} 0 & \dfrac{c}{\sqrt{\gamma}} & 0 & 0 \\ \dfrac{c}{\sqrt{\gamma}} & u & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}c \\ 0 & 0 & u & 0 \\ 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}c & 0 & u \end{bmatrix},$$

$$S_P^{-1}BS_P = \begin{bmatrix} v & 0 & \dfrac{c}{\sqrt{\gamma}} & 0 \\ 0 & v & 0 & 0 \\ \dfrac{c}{\sqrt{\gamma}} & 0 & v & \sqrt{\dfrac{\gamma-1}{\gamma}}c \\ 0 & 0 & \sqrt{\dfrac{\gamma-1}{\gamma}}c & v \end{bmatrix},$$

$$S_P^{-1}CS_P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{\lambda+2\mu}{\rho} & 0 & 0 \\ 0 & 0 & \dfrac{\mu}{\rho} & 0 \\ 0 & 0 & 0 & \dfrac{\gamma\mu}{P_r\rho} \end{bmatrix},$$

$$S_P^{-1}DS_P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & \dfrac{\lambda+2\mu}{\rho} & 0 \\ 0 & 0 & 0 & \dfrac{\gamma\mu}{P_r\rho} \end{bmatrix},$$

$$S_P^{-1}ES_P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\lambda+\mu}{\rho} & 0 \\ 0 & \dfrac{\lambda+\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{A.2.8}$$

A.III. *The Two-Dimensional Conservation-Form Equations and the Two-Dimensional Algorithm*

The two-dimensional conservation form which corresponds to (3.2) is

$$\frac{\partial \mathbf{H}}{\partial t} + \frac{\partial}{\partial x}(\mathbf{F}_H + \mathbf{F}_P + \mathbf{F}_M) + \frac{\partial}{\partial y}(\mathbf{G}_H + \mathbf{G}_P + \mathbf{G}_M) \tag{A.3.2}$$

with the fluxes given by

$$\mathbf{F}_{\mathrm{H}} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (e+p)\,u \end{bmatrix}, \qquad \mathbf{F}_{\mathrm{P}} = \begin{bmatrix} 0 \\ -(\lambda + 2\mu)\dfrac{\partial u}{\partial x} \\ -\mu\dfrac{\partial v}{\partial x} \\ -(\lambda + 2\mu)\,u\dfrac{\partial u}{\partial x} - \mu v\dfrac{\partial v}{\partial x} - k\dfrac{\partial T}{\partial x} \end{bmatrix},$$

$$\mathbf{F}_{\mathrm{M}} = \begin{bmatrix} 0 \\ -\lambda\dfrac{\partial v}{\partial y} \\ -\mu\dfrac{\partial u}{\partial y} \\ -\lambda u\dfrac{\partial v}{\partial y} - \mu v\dfrac{\partial u}{\partial y} \end{bmatrix}, \tag{A.3.3}$$

$$\mathbf{GH} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (e+p)\,v \end{bmatrix}, \qquad \mathbf{G}_{\mathrm{P}} = \begin{bmatrix} 0 \\ -\mu\dfrac{\partial u}{\partial y} \\ -(\lambda + 2\mu)\dfrac{\partial v}{\partial y} \\ -(\lambda + 2\mu)\,v\dfrac{\partial v}{\partial y} - \mu u\dfrac{\partial u}{\partial y} - k\dfrac{\partial T}{\partial y} \end{bmatrix}, \tag{A.3.4}$$

$$\mathbf{G}_{\mathrm{M}} = \begin{bmatrix} 0 \\ -\lambda\dfrac{\partial v}{\partial x} \\ -\mu\dfrac{\partial u}{\partial x} \\ -\mu u\dfrac{\partial v}{\partial x} - \lambda v\dfrac{\partial u}{\partial x} \end{bmatrix}.$$

Then the two-dimensional algorithm is constructed as

$$U^{n+2} = \{[L_x(\Delta t_x)\, L_y^{\tau_y}(\Delta t_y)\, L_{xy}(\Delta t_{xy})\, L_{xx}^{\sigma}(\Delta t_{xx})\, L_{yy}^{\sigma s^2}(\Delta t_{yy})] \cdot [L_{yy}^{\sigma s^2}(\Delta t_{yy})\, L_{xx}^{\sigma}(\Delta t_{xx})\, L_{xy}(\Delta t_{xy})\, L_y^{\tau_y}(\Delta t_y)\, L_x(\Delta t_x)]\} \cdot U^n, \tag{A.3.6}$$

with the same definitions as in Section III for τ_y, σ, S^2. The two-dimensional scheme implementation corresponding to Eqs. (3.8)–(3.14) is as follows:

$$U^{*}_{j,k} = U^{n}_{j,k} - \frac{\Delta t_x}{\Delta x}\,(F^{n}_{\mathrm{H}j+1,k} - F^{n}_{\mathrm{H}j,k}),$$

$$U^{**}_{j,k} = U^{*}_{j,k} - \frac{\Delta t_x}{\Delta x}\,(F^{*}_{\mathrm{H}j,k} - F^{*}_{\mathrm{H}j-1,k}),$$

$$U^{n+1}_{j,k} = \tfrac{1}{2}(U^{n}_{j,k} + U^{**}_{j,k}) \equiv L_x(\Delta t_x)\, U^{n}_{j,k}, \qquad \text{(A.3.8)}$$

also,

$$U^{*}_{j,k} = U^{n}_{j,k} - \frac{\Delta t_{xx}}{\Delta x}\,(F^{n}_{\mathrm{P}j+1,k} - F^{n}_{\mathrm{P}j,k}),$$

$$U^{**}_{j,k} = U^{*}_{j,k} - \frac{\Delta t_{xx}}{\Delta x}\,(F^{*}_{\mathrm{P}j,k} - F^{*}_{\mathrm{P}j-1,k}),$$

$$U^{n+1}_{j,k} = \tfrac{1}{2}(U^{n}_{j,k} + U^{**}_{j,k}) \equiv L_{xx}(\Delta t_{xx})\, U^{n}_{j,k}. \qquad \text{(A.3.9)}$$

and for the y-direction

$$U^{*}_{j,k} = U^{n}_{j,k} - \frac{\Delta t_y}{\Delta y}\,(G^{n}_{\mathrm{H}j,k+1} - G^{n}_{\mathrm{H}j,k}),$$

$$U^{**}_{j,k} = U^{*}_{j,k} - \frac{\Delta t_y}{\Delta y}\,(G^{*}_{\mathrm{H}j,k} - G^{*}_{\mathrm{H}j,k-1}),$$

$$U^{n+1}_{j,k} = \tfrac{1}{2}(U^{n}_{j,k} + U^{**}_{j,k}) \equiv L_y(\Delta t_y)\, U^{n}, \qquad \text{(A.3.10)}$$

followed by

$$U^{*}_{j,k} = U^{n}_{j,k} - \frac{\Delta t_{yy}}{\Delta y}\,(G^{n}_{\mathrm{P}j,k+1} - G^{n}_{\mathrm{P}j,k}),$$

$$U^{**}_{j,k} = U^{*}_{j,k} - \frac{\Delta t_{yy}}{\Delta y}\,(G^{*}_{\mathrm{P}j,k} - G^{*}_{\mathrm{P}j,k-1}),$$

$$U^{n+1}_{j,k} = \tfrac{1}{2}(U^{n}_{j,k} + U^{**}_{j,k}) \equiv L_{yy}(\Delta t_{yy})\, U^{n}, \qquad \text{(A.3.11)}$$

and finally for the mixed derivative operator

$$U^{*}_{j,k} = U^{n}_{j,k} - \frac{\Delta t_{xy}}{2\Delta x}\,(F^{n}_{\mathrm{M}j+1,k} - F^{n}_{\mathrm{M}j-1,k}) - \frac{\Delta t_{xy}}{2\Delta y}\,(G^{n}_{\mathrm{M}j,k+1} - G^{n}_{\mathrm{M}j,k-1}),$$

$$U^{**}_{j,k} = U^{*}_{j,k} - \frac{\Delta t_{xy}}{2\Delta x}\,(F^{*}_{\mathrm{M}j+1,k} - F^{*}_{\mathrm{M}j-1,k}) - \frac{\Delta t_{xy}}{2\Delta y}\,(G^{*}_{\mathrm{M}j,k+1} - G^{*}_{\mathrm{M}j,k-1}),$$

$$U^{n+1}_{j,k} = \tfrac{1}{2}(U^{n}_{j,k} + U^{**}_{j,k}) \equiv L_{xy}(\Delta t_{xy})\, U^{n}_{j,k}. \qquad \text{(A.3.12)}$$

In the above the instructions for using forward or backward differencing are the same as those given in Section III for the three-dimensional scheme. The above algorithm is stable under conditions similar to those given for the three-dimensional case, namely,

$$\Delta t_x \leqslant \frac{\Delta x}{|u| + c}, \qquad \Delta t_y \leqslant \frac{\Delta y}{|v| + c}, \qquad \Delta t_{xx} \leqslant \frac{(\Delta x)^2}{2(\gamma\mu/P_r\rho)},$$

$$\Delta t_{yy} \leqslant \frac{(\Delta y)^2}{2(\gamma\mu/P_r\rho)} \qquad \text{and} \qquad \Delta t_{xy} = \Delta t_x.$$

The choice for the maximum eigenvalue of C and D is indicated by the fact that $\gamma\mu/P_r \geqslant \lambda + 2\mu \geqslant \mu$ for most fluids under non-extreme flow conditions.

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